

# A Lagrangian Description of the Higher-Order Painlevé Equations

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## Abstract

We derive the Lagrangians of the higher-order Painlevé equations using Jacobi's last multiplier technique. Some of these higher-order differential equations display certain remarkable properties like passing the Painlevé test and satisfy the conditions stated by Juráš, (Acta Appl. Math. 66 (2001) 25–39), thus allowing for a Lagrangian description.

## 1 Introduction

The study of higher-order Painlevé equations is interesting from the mathematical point of view because of the possibility of existence of new transcendental functions beyond the six Painlevé transcendents. In addition such higher-order Painlevé often have interesting physical and mathematical applications. For example it is known that special solutions of equations for the Korteweg de Vries hierarchy which are used for describing water waves can be expressed via the higher-order Painlevé equations.

The first Painlevé hierarchy was first introduced in [1]. Thereafter many results were obtained in the analysis of the higher Painlevé equations. Scaling similarity solutions of three integrable PDEs namely the Sawada-Kotera,

fifth order KdV and Kaup-Kupershmidt equations were considered in [2] where it was shown that these fourth-order ordinary differential equations (ODEs) may be written as non-autonomous Hamiltonian equations for time dependent generalizations of integrable cases of the Hénon-Heiles system.

In [3] it was proved that higher-order members for the first and second Painlevé hierarchies do not have polynomial first integrals and that their solutions can determine new transcendental functions. Lax pairs for some equations of these hierarchies are presented in [4] and the Cauchy problem for equations of these hierarchies can be solved by an analogy with the Cauchy problem of the well known Painlevé equations. The Painlevé tests for higher-order Painlevé equations were demonstrated in [5–7].

In [8] two new hierarchies of nonlinear ODEs were introduced which were called the  $K_1$  and  $K_2$  hierarchies and which may be considered as new higher Painlevé hierarchies. The equations of these hierarchies have all the properties that are unique to the famous Painlevé equations.

Shimomura in [9] presented an interesting expression for the first Painlevé hierarchy which allows us to consider new properties of equations. Poles and  $\alpha$  - points of the meromorphic solutions of the first Painlevé hierarchy was studied by Shimomura in [10], where a lower estimate for the number of poles of meromorphic solution is also given.

In [11] instanton-type solutions and some leading expressions for the second member of the first hierarchy were constructed using multiple-scale analysis. Recently Mo in [12] has applied a twistor description of the similarity reductions to the case of the KdV hierarchy to obtain the twistor spaces of the Painlevé I and Painlevé II hierarchy. Dai and Zhang [13] have extended the results by Boutroux [14,15] for the first Painlevé equation to the case of the first Painlevé hierarchy. The authors have shown that there are solutions characterized by divergent asymptotic expansions near infinity in specified sectors of the complex plane for higher-order analogue of the first Painlevé equation.

Some important results connected with higher-order Painlevé equations were also obtained in the papers [16,17]. In [16] Claeys and Vanlessen proved the universality of the correlation kernel in a double scaling limit near singular edge points in random matrix models that were built out of functions associated with a special solution of the second member for the first Painlevé hierarchy. In [17] the authors established the existence of real solution of the fourth-order analogue of the Painlevé equation and obtained the solvability of an associated Riemann - Hilbert problem through the approach of a vanishing lemma and found additionally the asymptotics of solutions.

The Hamiltonian structure of the second Painlevé hierarchy was considered in [18]. Here the authors introduced new canonical coordinates and

obtained the Hamiltonian for evolutions. They also gave an explicit formulae for these Hamiltonians and demonstrated that these Hamiltonians are polynomials in the canonical coordinates.

The aim of this paper is to obtain the Lagrangians for the four higher Painlevé hierarchies using the same approach. In recent years much attention has been paid to the Lagrangian framework of higher-order differential equations. Although a Lagrangian always exists for any second-order ordinary differential equation its connection with Jacobi's last multiplier (JLM) [19,20] is perhaps not very widely known. The credit for resurrecting the JLM, in recent years, must go to Leach and Nucci, who have shown how it may be used to determine the first integrals and also Lagrangians of a wide variety of nonlinear differential equations [21]. While it appears that the connection of the Jacobi last multiplier to the existence of Lagrangian functions were the subjects of investigation by a few authors in the early 1900's, the precise nature of this interrelation was brought out by Rao, in the 1940's [22]. Thereafter it does not appear to have attracted the attention of most researchers working in the field of differential equations.

According to the classical theory of Darboux [23] every scalar second-order ordinary differential equation is multiplier variational. The problem of finding a Lagrangian for a given ODE is generally referred to as the inverse variational problem of classical mechanics. The necessary and sufficient conditions for an equation  $y'' = F(x, y, y')$  to be derivable from the Euler-Lagrange equation  $\frac{\partial L}{\partial y} - \frac{d}{dx} \left( \frac{\partial L}{\partial y'} \right) = 0$ , was enunciated by Helmholtz [24, 25] in the form of certain identities.

The variational multiplier problem for higher-order scalar ordinary differential equations has been studied by Fels [26] and Juráš [27]. The inverse problem for a fourth-order ODE was solved by Fels who investigated scalar fourth-order ordinary differential equations of the form

$$\frac{d^4 u}{dx^4} = f\left(x, u, \frac{du}{dx}, \frac{d^2 u}{dx^2}, \frac{d^3 u}{dx^3}\right).$$

Fels approach for solving the fourth-order inverse problem was essentially based on a modified version of Douglas's [28] classical solution to the multiplier problem as refined by Anderson and Thompson in [29], who used the variational bicomplex theory [30] to derive the multiplier and showed that the existence of a multiplier was in a direct correspondence with the existence of special cohomology classes arising in the variational bicomplex associated with a differential equation. Fels conditions ensure the existence and uniqueness of the Lagrangian in the case of a fourth-order equation and it has been shown by Nucci and Arthurs [31] and more recently by us [32] that when

these conditions are satisfied, a Lagrangian can be derived from the Jacobi last multiplier.

In fact Fels approached the problem using Cartan's equivalence method, and arrived at two differential invariants whose vanishing completely characterizes the existence of a variational multiplier. Unlike the second-order case, the multiplier is unique up to a constant multiple. The programme was further developed by Juráš [27] who studied the inverse problem for sixth and eighth-order equations. In fact Juráš obtained a similar solution by using, however, a more direct approach in the spirit of the variational bicomplex; the differential invariants becoming increasingly complicated for higher-order systems. By analyzing the structure equations of the horizontal differential he uncovered a two-form  $\Pi$  with the property  $d\Pi \equiv 0 \pmod{\Pi}$ , if and only if the equation

$$\frac{d^{2n}u}{dx^{2n}} = f\left(x, u, \frac{du}{dx}, \dots, \frac{d^{2n-1}u}{dx^{2n-1}}\right),$$

is multiplier variational. He proved that a Lagrangian, if it exists, is unique up to the multiplication by a constant and found functions  $I_1, I_2, \dots, I_n$ , whose vanishing provides a necessary condition for the above equation to be variational. These functions are not relative contact invariants, but their simultaneous vanishing is a contact invariant condition.

In [32] the authors made use of the Jacobi Last Multiplier (JLM) to derive Lagrangians for a set of fourth-order ODEs which pass the Painlevé test, i.e., their solutions are free of movable critical points. Recently the conjugate Hamiltonian equations for such fourth-order equations passing the Painlevé test have also been derived in [33].

## 2 Four Painlevé hierarchies

Now the first and the second Painlevé hierarchy are well known and can be written as the following

$$\sum_{m=1}^N t_m L_m[u] = x, \tag{1}$$

$$\left(\frac{d}{dx} + u\right) \sum_{m=1}^M t_m L_m[u_x - u^2] - x u - \beta_N = 0, \tag{2}$$

where  $N$  and  $M$  are integers,  $t_m$ , ( $m = 1, \dots, N$ ) is the sequence of operators  $L_m[u]$  that satisfies the Lenard recursion relation

$$d_x L_{m+1}[u] = (d_x^3 + 4u d_x + 2u_x) L_m[u], \quad L_0[u] = \frac{1}{2}. \tag{3}$$

Taking the operator (3) into account we obtain

$$L_1[u] = u, \quad (4)$$

$$L_2[u] = u_{xx} + 3u^2, \quad (5)$$

$$L_3[u] = u_{xxxx} + 10u u_{xx} + 5u_x^2 + 10u^3, \quad (6)$$

$$L_4[u] = u_{xxxxx} + 14u u_{xxx} + 28u_x u_{xx} + 21u_{xx}^2 + 70u^2 u_{xx} + 70u u_x^2 + 35u^4. \quad (7)$$

Using the values of operators  $L_1, L_2, L_3, L_4$  and so on we can obtain the equations of the first and the second Painlevé hierarchies.

The sixth-order ordinary differential equations of the first and the second Painlevé hierarchies have the form

$$\begin{aligned} & t_4 (u_{xxxxx} + 14u u_{xxx} + 28u_x u_{xx} + 21u_{xx}^2 + 70u^2 u_{xx} + \\ & + 70u u_x^2 + 35u^4) + t_3 (u_{xxx} + 10u u_{xx} + 5u_x^2 + 10u^3) + \\ & + t_2 (u_{xx} + 3u^2) + t_1 u = x, \end{aligned} \quad (A)$$

$$\begin{aligned} & t_3 (u_{xxxxx} - 14u^2 u_{xxx} - 56u u_x u_{xx} - 28u_x^2 u_{xx} - 42u u_{xx}^2 + \\ & + 70u^4 u_{xx} + 140u^3 u_x^2 - 20u^7) + t_2 (u_{xxx} - 10u^2 u_{xx} - \\ & - 10u u_x^2 + 6u^5) + t_1 (u_{xx} - 2u^3) - xu - \beta_3 = 0, \end{aligned} \quad (B)$$

We see that equations of the first and the second hierarchy have even integer orders  $2N - 2$  and  $2M$  respectively.

Equations (A) and (B) are important and interesting because setting the constants  $t_3 = t_2 = 0$  one recovers the Painlevé equations. When  $t_1 = t_3 = 0$  these yield equations which we have studied recently. In the case  $t_1 = t_2 = 0$  they reduce to sixth-order equations which are the third members of the first and second Painlevé hierarchies. The general case of these equations correspond to the first and second Painlevé hierarchies.

There are two other hierarchies of nonlinear ordinary differential equations that have the properties similar to Painlevé equations. These hierarchies were introduced in [8] and were referred to in [34] as the  $K_1$  and  $K_2$  hierarchies. These hierarchies can be presented as the following

$$\sum_{m=1}^N t_m H_m[u] = x, \quad (8)$$

$$\left(\frac{d}{dx} + u\right) \sum_{m=1}^M t_m H_m \left[ u_x - \frac{1}{2} u^2 \right] - x u - \beta_M = 0, \quad (9)$$

where  $N$  and  $M$  are integers,  $t_m$  are parameters of the equation and the operator  $H_m$  may be calculated by means of the formulae

$$H_{n+2} = J[v] \Omega[v] H_n, \quad (10)$$

under the conditions

$$H_0[v] = 1, \quad H_1[v] = v_{xx} + 4v^2, \quad (11)$$

where the operators  $\Omega[v]$  and  $J[v]$  are determined by the relations

$$\Omega = D^3 + 2vD + v_x, \quad D = \frac{d}{dx}, \quad (12)$$

$$\begin{aligned} J = D^3 + 3(vD + Dv) + 2(D^2vD^{-1} + D^{-1}vD^2) + \\ + 8(v^2D^{-1} + D^{-1}v^2), \quad D^{-1} = \int dx. \end{aligned} \quad (13)$$

Taking conditions (11) and operators (12), (13) into account we have the operators  $H_2$  and  $H_3$  as the following

$$H_2[v] = v_{xxxx} + 12v v_{xx} + 6v_x^2 + \frac{32}{3}v^3, \quad (14)$$

$$\begin{aligned}
H_3[v] = & v_{xxxxxxx} + 20 v v_{xxxxxx} + 60 v_x v_{xxxxx} + 134 v_{xx} v_{xxxx} + \\
& + 136 v^2 v_{xxx} + 84 v_{xxx}^2 + 544 v v_x v_{xxx} + 408 v v_{xx}^2 + 396 v_x^2 v_{xx} + \\
& + \frac{1120}{3} v^3 v_{xx} + 560 v^2 v_x^2 + \frac{256}{3} v^5.
\end{aligned} \tag{15}$$

Note that hierarchies (8) and (9) can also be presented using another operator  $G_k[u]$ . In terms of this operator these hierarchies take in the form

$$\sum_{k=1}^N t_k G_k[u] = x. \tag{16}$$

$$\left(u - \frac{1}{2} \frac{d}{dx}\right) \sum_{k=1}^M t_k G_k[-2u_x - 2u^2] - xu - \beta_M = 0. \tag{17}$$

Hierarchy (16) can be transformed to (8) but hierarchy (17) coincides with hierarchy (9). The recursion relation  $G_k$  is determined by the nonlinear operator

$$G_{k+2} = J_1[v] \Omega[v] G_k, \tag{18}$$

under the conditions

$$G_0[v] = 1, \quad G_1[v] = v_{xx} + \frac{1}{4} v^2. \tag{19}$$

The operator  $J_1[v]$  takes the form

$$J_1 = D^3 + \frac{1}{2} (D^2 v D^{-1} + D^{-1} v D^2) + \frac{1}{8} (v^2 D^{-1} + D^{-1} v^2). \tag{20}$$

Hierarchies  $K_1$  and  $K_2$  though similar to the first and the second Painlevé hierarchies have a fundamental difference in the sense that we cannot transform equations of hierarchies (16) and (17) to hierarchies (1) and (2). Moreover the hierarchy  $K_1$  has even integer order except  $6k$  ( $k = 1, 2, \dots$ ) and hierarchy  $K_2$  also has even integer order except  $6k$  ( $k = 1, 2, \dots$ ).

The fourth order equation corresponding to the hierarchy  $K_1$  takes the form

$$t_2 \left( u_{xxxx} + 12 u u_{xx} + 6 u_x^2 + \frac{32}{3} \right) + t_1 (u_{xx} + 4 u^2) = 0. \quad (C)$$

At  $t_2 = 0$  equation (C) is the first Painlevé equation but at  $t \neq 0$  the forth order equation differs from the the fourth order equation of the first Painlevé equation and we hope that this one may give a new transcendal function.

On the other hand the sixth-order equation from hierarchy  $K_2$  can be written as

$$\begin{aligned} t_2 \left( u_{xxxxx} + 7 u_x u_{xxx} - 7 u^2 u_{xxx} + 14 u_{xx} u_{xx} - 28 u u_x u_{xx} - \right. \\ \left. - 28 u_x^2 u_{xx} - 21 u u_{xx}^2 - \frac{28}{3} u u_x^3 - 14 u^2 u_x u_{xx} + 14 u^4 u_{xx} + \right. \\ \left. + 28 u^3 u_x^2 - \frac{4}{3} u^7 \right) + t_1 (u_{xxxx} + 5 u_x u_{xx} - 5 u^2 u_{xx} - \\ - 5 u u_x^2 + u^5) - x u - \beta_2 = 0. \end{aligned} \quad (D)$$

Equation (D) is a sixth-order nonlinear ordinary differential equation with properties similar to the Painlevé equations but cannot be transformed to the equation of the second Painlevé hierarchy. This equation does not have a first integral in the polynomial form and it is possible that it determines a new transcendental function.

In the following section we find the Lagrangians for the nonlinear ordinary differential equations (A), (B) and (D).

### 3 Inverse problem for sixth-order equations and their Lagrangians

Consider a sixth-order equation in the normal form,  $u_6 = f(x, u, u_1, u_2, u_3, u_4, u_5)$ . Here we introduce the abridged notation  $u_k = d^k u / dx^k$ . The following theorem due to Juráš gives the necessary and sufficient conditions for a sixth-order equation to admit a variational multiplier [27].

**Theorem.** A sixth-order ordinary differential equation admits a variational multiplier and non-degenerate third-order Lagrangian if and only if following two conditions are satisfied

$$0 = -\frac{2}{3} D_x^4 \left( \frac{\partial f}{\partial u_5} \right) + \frac{10}{9} \frac{\partial f}{\partial u_5} D_x^3 \left( \frac{\partial f}{\partial u_5} \right) + D_x^3 \left( \frac{\partial f}{\partial u_4} \right) + \frac{20}{9} D_x \left( \frac{\partial f}{\partial u_5} \right) D_x^2 \left( \frac{\partial f}{\partial u_5} \right)$$



$$\begin{aligned}
& -\frac{20}{27}\left(\frac{\partial f}{\partial u_5}\right)^2 D_x^2\left(\frac{\partial f}{\partial u_5}\right) - \frac{1}{3}\frac{\partial f}{\partial u_4} D_x^2\left(\frac{\partial f}{\partial u_5}\right) - \frac{\partial f}{\partial u_5} D_x^2\left(\frac{\partial f}{\partial u_4}\right) - D_x^2\left(\frac{\partial f}{\partial u_3}\right) \\
& - \frac{10}{9}\frac{\partial f}{\partial u_5} \left(D_x\left(\frac{\partial f}{\partial u_5}\right)\right)^2 - D_x\left(\frac{\partial f}{\partial u_5}\right) D_x\left(\frac{\partial f}{\partial u_4}\right) + \frac{20}{81}\left(\frac{\partial f}{\partial u_5}\right)^3 D_x\left(\frac{\partial f}{\partial u_5}\right) \\
& + \frac{1}{3}\left(\frac{\partial f}{\partial u_5}\right)^2 D_x\left(\frac{\partial f}{\partial u_4}\right) + \frac{1}{3}\frac{\partial f}{\partial u_5}\frac{\partial f}{\partial u_4} D_x\left(\frac{\partial f}{\partial u_5}\right) + \frac{1}{3}\frac{\partial f}{\partial u_3} D_x\left(\frac{\partial f}{\partial u_5}\right) + \frac{2}{3}\frac{\partial f}{\partial u_5} D_x\left(\frac{\partial f}{\partial u_3}\right) \\
& + D_x\left(\frac{\partial f}{\partial u_2}\right) - \frac{2}{243}\left(\frac{\partial f}{\partial u_5}\right)^5 - \frac{1}{27}\left(\frac{\partial f}{\partial u_5}\right)^3 \frac{\partial f}{\partial u_4} - \frac{1}{9}\left(\frac{\partial f}{\partial u_5}\right)^2 \frac{\partial f}{\partial u_3} - \frac{1}{3}\frac{\partial f}{\partial u_5}\frac{\partial f}{\partial u_2} - \frac{\partial f}{\partial u_1},
\end{aligned}$$

and

$$\begin{aligned}
0 = & \frac{5}{3}D_x^2\left(\frac{\partial f}{\partial u_5}\right) - \frac{5}{3}\frac{\partial f}{\partial u_5} D_x\left(\frac{\partial f}{\partial u_5}\right) - 2D_x\left(\frac{\partial f}{\partial u_4}\right) + \\
& \frac{5}{27}\left(\frac{\partial f}{\partial u_5}\right)^3 + \frac{2}{3}\frac{\partial f}{\partial u_5}\frac{\partial f}{\partial u_4} + \frac{\partial f}{\partial u_3}.
\end{aligned}$$

**Prove.** Suppose the sixth-order equation

$$u_6 = f(x, u, u_1, u_2, u_3, u_4)$$

is independent of  $u_5$ . Then it admits a variational multiplier and a non-degenerate third-order Lagrangian if and only if the following two conditions are satisfied:

$$0 = D_x^3\left(\frac{\partial f}{\partial u_4}\right) - D_x^2\left(\frac{\partial f}{\partial u_3}\right) + D_x\left(\frac{\partial f}{\partial u_2}\right) - \frac{\partial f}{\partial u_1} \text{ and } 0 = -2D_x\left(\frac{\partial f}{\partial u_4}\right) + \frac{\partial f}{\partial u_3}.$$

### 3.1 The Jacobi Last Multiplier and construction of Lagrangians for sixth-order equations

In this section we describe the connection of the Jacobi Last Multiplier with the Lagrangian function for sixth-order ODEs.

**Proposition.** Let  $u_6 = f(x, u, u_1, u_2, u_3, u_4, u_5)$  be a sixth-order ordinary differential equation which admits a Lagrangian  $L$ . Then the function  $\mathcal{M} := \left(\frac{\partial^2 L}{\partial u_3^2}\right)^3$ , is a Jacobi last multiplier, i.e., it satisfies the equation  $\frac{d\mathcal{M}}{dx} + \frac{\partial f}{\partial u_5}\mathcal{M} = 0$ , where  $u_5 = u_{xxxxx}$ .

**Proof :** Considering the higher-order Euler operator,  $E$ , the Euler-Lagrange equation of motion for the ODE  $u_6 = f(x, u, u_1, \dots, u_5)$  is given by

$$E(L) = \frac{\partial L}{\partial u} - D_x\left(\frac{\partial L}{\partial u_1}\right) + D_x^2\left(\frac{\partial L}{\partial u_2}\right) - D_x^3\left(\frac{\partial L}{\partial u_3}\right) = 0, \quad (21)$$

where  $L = L(x, u, u_1, u_2, u_3)$  is a third-order Lagrangian. It is obvious that the partial derivatives of  $L$ , namely  $L_u, L_{u_1}, \dots, L_{u_3}$  are all functions of  $x, u, \dots, u_3$ . Upon expanding the Euler-Lagrange equation we find that

$$0 = E(L) = u_5 L_{u_2 u_3} - [2u_5 L_{u_3 u_3 x} + u_5 L_{u_2 u_3} + f(x, u, u_1, \dots, u_5) L_{u_3 u_3} + u_5 D_x(L_{u_3 u_3}) + 2u_4 u_5 L_{u_3 u_3 u_3} + 2u_5 u_1 L_{u_3 u_3 u} + 2u_2 u_5 L_{u_3 u_3 u_1} + 2u_3 u_5 L_{u_3 u_3 u_2}] + \text{terms independent of } u_5.$$

Here the subscripts denote partial derivatives with respect to the indicated variables. Since the partial derivative of this equation with respect to  $u_5$  must also be identically zero, we find that  $3D_x(L_{u_3 u_3}) + \frac{\partial f}{\partial u_5}(L_{u_3 u_3}) = 0$ .

Let be  $M_{(3)} = L_{u_3 u_3}$ , then the above equation,  $E(L) = 0$  is expressible as be  $D_x(\log M_{(3)}^3) + \frac{\partial f}{\partial u_5} = 0$ , showing thereby that the Jacobi Last multiplier is given by

$$\mathcal{M} = M_{(3)}^3. \quad \square$$

**Remark:** Note that for the fourth-order ODE,  $u_4 = f(x, u, \dots, u_3)$ , admitting a second-order Lagrangian the analog of (3.1) is the following equation:

$$D_x(\log M_{(2)}^2) + \frac{\partial f}{\partial u_3} = 0,$$

so that the JLM is  $\mathcal{M} = M_{(2)}^2$  where  $M_{(2)} = L_{u_2 u_2}$ . On the other hand for the second-order ODE,  $u_2 = f(x, u, u_1)$ , it is the solution of

$$D_x(\log M_{(1)}) + \frac{\partial f}{\partial u_1} = 0,$$

with  $\mathcal{M} = M_{(1)} = L_{u_1 u_1}$ .

Equation (??) provides us a tool for determining the Lagrangian of a fourth-order equation once a solution of the defining equation for the JLM,  $\mathcal{M}$ , is obtained from (??). In fact in the event  $f$  is independent of  $u_5$ , so that the condition (??) is trivially satisfied one obtains the solution  $M_{(3)} = \text{constant}$ , which may be set equal to unity, without loss of generality. In such a situation the Juráš conditions are also considerably simplified as evident from the Corollary 1.

### 3.2 Determination of the Lagrangians

We wish to determine a nondegenerate third-order Lagrangian  $L = L(x, u, u_1, u_2, u_3)$  such that  $E(L) = 0$ , where  $\frac{\partial^2 L}{\partial u_3^2} \neq 0$ , where  $E$  is the Euler-Lagrange operator  $E = \frac{\partial}{\partial u} - D_x\left(\frac{\partial}{\partial u_1}\right) + D_x^2\left(\frac{\partial}{\partial u_2}\right) - D_x^3\left(\frac{\partial}{\partial u_3}\right)$ , and  $D_x$  denotes the total derivative operator  $D_x = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + u_3 \frac{\partial}{\partial u_2}$ . If there is a third-order Lagrangian satisfying the conditions stated in Theorem 3.1, one says

that the ordinary differential equation  $u_6 = f(x, u, u_1, u_2, u_3, u_4, u_5)$  admits a variational multiplier.

In the new notation equation (A) is given by

$$t_3(u_6 + 14uu_4 + 28u_1u_3 + 21u_2^2 + 70u^2u_2 + 70uu_1^2 + 35u^4) + t_2(u_4 + 10uu_2 + 5u_1^2 + 10u^3) + t_1(u_2 + 3u^2) = x.$$

**Proposition.** Equation (A) admits a Lagrangian description with Lagrangian  $L = t_3(-\frac{1}{2}u_3^2 + 7u^5 - 35u^2u_1^2 + 7uu_2^2) + t_2(\frac{1}{2}u_2^2 - 5uu_1^2 + \frac{5}{2}u^4) + t_1(-\frac{1}{2}u_1^2 + u^3) - x u$ , where  $u_k = u^{(k)}$ ,  $k = 1, 2, \dots$ .

**Outline of the proof :** In order to show this we will adopt the technique used in [32], to derive Lagrangians for a certain class of fourth-order ODEs, namely that of the Jacobi Last Multiplier (JLM). For a sixth-order ODE written in the form

$$u_6 = f(x, u, u_1, \dots, u_5)$$

one can rewrite the equation as a first-order system:  $u_1 = v$ ,  $v_1 = w$ ,  $w_1 = s$ ,  $s_1 = t$ ,  $t_1 = r$ ,  $r_1 = f(x, u, v, w, s, t, r)$ , where the subscript 1 denotes differentiation with respect to the independent variable  $x$ . Then by definition the JLM,  $\mathcal{M}$ , for the above system of first-order ODEs is defined as the solution of the following equation  $\frac{d \log \mathcal{M}}{dx} + \left( \frac{\partial v}{\partial u} + \frac{\partial w}{\partial v} + \frac{\partial s}{\partial w} + \frac{\partial t}{\partial s} + \frac{\partial r}{\partial t} + \frac{\partial f}{\partial r} \right) = 0$ . Since in case of eqns (A) and (B) the function  $f$  is independent of  $u_5$  i.e., of  $r$  in this notation it follows that  $\frac{d \log M_{(3)}}{dx} = 0 \Rightarrow M_{(3)} = \text{constant}$ . Furthermore since the JLM is connected to the Lagrangian by the following relation  $M_{(3)} = \frac{\partial^2 L}{\partial u_3^2}$ , where  $u_3 = \frac{d^3 u}{dx^3}$ , therefore setting the constant in (??) to be  $-t_3$  we have

$$\frac{\partial^2 L}{\partial u_3^2} = -t_3 \Rightarrow L = -t_3 \frac{u_3^2}{2} + R(x, u, u_1, u_2)u_3 + V(x, u, u_1, u_2).$$

Consequently the determination of the Lagrangian essentially reduces to finding appropriate functions  $R$  and  $V$  such that the Euler-Lagrange equation  $E(L) = 0$ , reproduces the desired equation, namely (??). Detailed calculations however show that it is possible to choose  $R = 0$  so as to simplify the resulting calculations, and therefore the Lagrangian is of the form,  $L_A = t_3(-\frac{1}{2}u_3^2) + V(x, u, u_1, u_2)$ . To deduce the unknown function  $V$  we substitute this form of the Lagrangian into (21) and compare the resulting equation with our original sixth-order ODE. One finds that in this case

$$E(L_A) = \frac{\partial L_A}{\partial u} - D_x \left( \frac{\partial L_A}{\partial u_1} \right) + D_x^2 \left( \frac{\partial L_A}{\partial u_2} \right) - D_x^3 \left( \frac{\partial L_A}{\partial u_3} \right) = 0$$

gives

$$-t_3 u_6 = V_u - V_{xu_1} - u_1 V_{uu_1} - u_2 V_{u_1 u_1} + u_4 V_{u_2 u_2} + V_{xx u_2} + 2u_1 V_{xu u_2} + 2u_2 V_{xu_1 u_2} + 2u_3 V_{xu_2 u_2}$$

$$+u_1^2 V_{uuu_2} + u_2^2 V_{u_1 u_1 u_2} + u_3^2 V_{u_2 u_2 u_2} + 2u_1 u_2 V_{uu_1 u_2} + 2u_1 u_3 V_{uu_2 u_2} + 2u_2 u_3 V_{u_1 u_2 u_2}.$$

Inserting the value of  $u_6$  from the original equation and equating the coefficient of  $u_4$  we find that

$$V_{u_2 u_2} = t_3 14u + t_2$$

which leads to the following solution, namely  $V = t_3 7uu_2^2 + t_2 \frac{u_2^2}{2} + Tu_2 + S$  where  $T$  and  $S$  are functions of  $x, u, u_1$ . Once again we may set  $T = 0$  so that  $L_A = t_3 \left(-\frac{1}{2}u_3^2 + 7uu_2^2\right) + t_2 \left(\frac{1}{2}u_2^2\right) + S$ , and it remains therefore to determine the unknown function  $S$ . From the remaining terms we find that one must have  $S_u - S_{xu_1} - u_1 S_{uu_1} - u_2 S_{u_1 u_1} = t_3(70u^2 u_2 + 70uu_1^2 + 35u^4) + t_2(10uu_2 + 5u_1^2 + 10u^3) + t_1(u_2 + 3u^2) - x$ . Next equating the coefficients of  $u_2$  it is seen that  $-S = t_3 35u^2 u_1^2 + t_2 5uu_1^2 + t_1 \frac{u_1^2}{2} + Ku_1 + N(x, u)$ . Again choosing  $K = 0$ , we have ultimately from (3.2),

$$-N_u = t_3 35u^4 + t_2 10u^3 + t_1 3u^2 - x.$$

This yields

$$-N = t_3 7u^5 + t_2 \frac{5}{2}u^4 + t_1 u^3 - xu,$$

and finally the following expression for the unknown function  $S$

$$S = -t_3 35u^2 u_1^2 - t_2 5uu_1^2 + t_1 \left(-\frac{1}{2}u_1^2\right) + t_3 7u^5 + t_2 \frac{5}{2}u^4 + t_1 u^3 - xu.$$

We find finally that the expression for the Lagrangian of eqn. (A) is,

$$L_A = t_3 \left(-\frac{1}{2}u_3^2 + 7u^5 - 35u^2 u_1^2 + 7uu_2^2\right) + t_2 \left(\frac{1}{2}u_2^2 - 5uu_1^2 + \frac{5}{2}u^4\right) + t_1 \left(-\frac{1}{2}u_1^2 + u^3\right) - xu.$$

□

In a similar manner we find that the sixth-order equations (B) and (D) also admit a Lagrangian description, which are stated below.

**Proposition.** The Lagrangians associated with the equations (B) and (D) are given by

$$\begin{aligned} L_B = & t_1 \left(-\frac{1}{2}u_1^2 - \frac{1}{2}u^4\right) + t_2 \left(\frac{1}{2}u_2^2 + 5u^2 u_1^2 + u^6\right) + \\ & + t_3 \left(-\frac{1}{2}u_3^2 - \frac{5}{2}u^8 - 35u^4 u_1^2 - 7u^2 u_2^2\right) - \frac{1}{2}xu^2 - \beta_3 u, \end{aligned} \quad (22)$$

$$\begin{aligned} L_D = & t_2 \left(-\frac{1}{2}u_3^2 + \frac{7}{2}u_1 u_2^2 - \frac{7}{2}u^2 u_2^2 + \frac{7}{6}u_1^4 + \frac{7}{3}u^2 u_1^3 - 7u^4 u_1^2 - \frac{u^8}{6}\right) + \\ & + t_1 \left(\frac{1}{2}u_2^2 + \frac{5}{2}u^2 u_1^2 - \frac{5}{6}u_1^3 + \frac{u^6}{6}\right) - \frac{1}{2}xu^2 - \beta_2 u. \end{aligned} \quad (23)$$

## 4 Conclusion

Let us briefly discuss the results of this paper. We have found the Lagrangians  $L_A$ ,  $L_B$  and  $L_D$  for the sixth-order nonlinear differential equations from the Painlevé hierarchies. These Lagrangians are generalizations of the well known Lagrangians of the Painlevé equations. It is interesting to look at the properties of the Lagrangians  $L_A$ ,  $L_B$  and  $L_D$ . For example we know that there exists the following symmetry of equation (B) when  $u(x) \rightarrow -u(x)$  and  $\beta_3 \rightarrow -\beta_3$ . This symmetry is preserved for the Lagrangian  $L_B$  as well. Equations (A), (B) and (D) possess the Painlevé property. The Lagrangians of several mechanical systems usually involve a difference of their kinetic and potential energies respectively. Here this property holds for the Lagrangians  $L_A$  and  $L_B$  at  $t_2 = t_3 = x = 0$ . However this is not true in the general case and as such the Lagrangians derived here are really examples of nonstandard ones. The question of irreducibility of the higher Painlevé equations is an open problem.

## Acknowledgement

We wish to thank Pepin Cariñena, Thanasis Fokas, Peter Leach and Mark Fels for their valuable comments. One of us (AGC) wishes to acknowledge the support provided by the S. N. Bose National Centre for Basic Sciences, Kolkata in the form of an Associateship.

## References

- [1] N.A. Kudryashov, *The first and second Painlevé equations of higher order and some relations between them*, Physics Letters A, 224 (1997) 353 – 360
- [2] A.N.W. Hone, *Nonautonomous Hénon-Heiles systems*, Physica D 118 (1998) 1 - 16
- [3] N.A. Kudryashov, *On transcendents defined by nonlinear ordinary differential equations*, (1998), J.Phys. A: Math.Gen. **31** L129.
- [4] N.A. Kudryashov, M.B. Soukharev, *Uniformization and transcendence of solutions for the first and second Painlevé hierarchies*, Phys. Lett. A 237(4-5) (1998) 206-216.

- [5] U. Mugan, F. Jrad, *Painleve test and the first Painlevé hierarchy*, J. Phys. A: Math. Gen. 32 (1999) 7933
- [6] N.A. Kudryashov, *Some fourth-order ordinary differential equations which pass the Painlevé test*. J. Nonlinear Math. Phys. 8 (2001), 172-177.
- [7] U. Mugan, F. Jrad, *Painlevé test and Higher order differential equations*, J. Nonlinear Math. Phys. 9(3) (2002) 1- 29
- [8] N.A. Kudryashov, *Two hierarchies of ordinary differential equations and their properties*, Physics Letters A, 252 (1999) 173 – 179.
- [9] S. Shimomura, *A certain expression for the first Painleve hierarchy*, Proc. Japan Acad. Ser. A 80 (2004) 105 - 109.
- [10] S. Shimomura, *Poles and  $\alpha$ -points of meromorphic solutions of the first Painlevé hierarchy*, Publ. RIMS, 40 (2004) 471 - 485.
- [11] T. Aoki, *Multiple-Scale analysis for higher-order Painlevé equations*, RIMS, B5 (2008) 89 - 08
- [12] M.Y. Mo, *Twistor reductions of the KdV hierarchy*, Journal of Geometry and Physics 59 (2009) 234-245.
- [13] D. Dai, L. Zhang, *On troquée solutions of the first Painlevé hierarchy*, J. Math. Anal. Appl. 368 (2010) 393 - 399.
- [14] P. Boutroux, *Recherches sur les transcendentes de M. Painlevé et l'étude asymptotique des équations différentielles du second ordre*, Ann. Sci. École Norm. Sup. (3) 30 (1913), 255–375.
- [15] P. Boutroux, *Recherches sur les transcendentes de M. Painlevé et l'étude asymptotique des équations différentielles du second ordre (suite)*, Ann. Sci. École Norm. Sup. (3) 31 (1914), 99–159.
- [16] T. Claeys, M. Vanlessen, *The existence of a real pole - free solution of the fourth order analogue of the Painlevé I equation*, Nonlinearity 20 (2007) 1163 - 1184
- [17] T. Claeys, M. Vanlessen, *Universality of a double scaling limit near singular edge points in random matrix models*, Comm. Math. Phys. 273 (2007) 499 - 532
- [18] M. Mazzocco, M.Y. Mo, *The Hamiltonian structure of the second Painleve hierarchy*, Nonlinearity 20 (2007) 2845 - 2882

- [19] C.G.J. Jacobi, *Sul principio dell'ultimo moltiplicatore, e suo uso come nuovo principio generale di meccanica*, Giornale Arcadico di Scienze, Lettere ed Arti **99** (1844), 129-146.
- [20] C.G.J. Jacobi, *Theoria novi multiplicatoris systemati aequationum differentialium vulgarium applicandi*, J. Reine Angew. Math **27** (1844), 199-268, *Ibid* **29**(1845), 213-279 and 333-376.
- [21] M.C. Nucci and P.G.L. Leach *The method of Ostrogradsky, quantization, and a move toward a ghost-free future*, J. Math. Phys. **50**, 113508 (2009) 6 pp.
- [22] B.S. Madhava Rao, *On the reduction of dynamical equations to the Lagrangian form*. Proc. Benaras Math. Soc. (N.S.) 2, (1940). 53–59.
- [23] G. Darboux, *Lecon sur la théorie générale des surfaces* . Vol. III, Gauthier-Villar, Paris 1894.
- [24] H. Helmholtz, *Journ. Reine und Angewandte Mathematik* **100**, (1887), 137.
- [25] Jan Lopuszanski, *The Inverse Variational Problem in Classical Mechanics*, World Scientific, Singapore, 1999.
- [26] M.E. Fels, *The inverse problem of the calculus of variations for scalar fourth-order ordinary differential equations*, Trans. Amer. Math. Soc. **348** (1996), 5007-5029.
- [27] M. Juráš, *The inverse problem of the calculus of variations for sixth- and eighth-order scalar ordinary differential equations*. Acta Appl. Math. **66** (2001), no. 1, 25–39.
- [28] J. Douglas, *Solution to the inverse problem of the calculus of variations*, Trans. Amer. Math. Soc. **50** (1941) 71-128.
- [29] I. Anderson and G. Thompson, *The inverse problem of the calculus of variations for ordinary differential equations*. Mem. Amer. Math. Soc. **98** (1992), no. 473,
- [30] I. Anderson, *Introduction to the variational bicomplex*, [www.math.usu.edu/~fgmp/.../IntroVariationalBicomplex.pdf](http://www.math.usu.edu/~fgmp/.../IntroVariationalBicomplex.pdf)
- [31] M.C Nucci and A. M. Arthurs, *On the inverse problem of calculus for fourth-order equations*, Proc.R. Soc. A doi: 10.1098/rspa.2009.0618.

- [32] P. Guha and A. Ghose Choudhury, *On Lagrangians and Hamiltonians of Some Fourth-Order Nonlinear Kudryashov ODEs*, Comm. Non. Sc. Num. Sim. 16 (2011) 3914-3922.
- [33] P. Guha, A. Ghose Choudhury and A.S. Fokas, *Hamiltonians and Conjugate Hamiltonians of Some Fourth-Order Nonlinear ODEs*, Preprint.
- [34] N.A. Kudryashov, *Double Bäcklund transformations and special integrals for the  $K_{II}$  hierarchy*, Physics Letters A 273 (2000) 194-202